Resit Probability Theory (WBMA046-05) Solutions

13 July 2022, 16:00-18:00, Exam Hall 1 H1 - J7

READ THE INSTRUCTIONS BELOW CAREFULLY BEFORE STARTING THE EXAM.

- This exam contains 6 pages (including this cover page) and 5 exercises.
- Write your name and student number at the top of EACH page (including this cover page).
- Your answers should be written in this booklet. Preferably, avoid handing in extra paper.
  If you do not have enough space to write your solution below the exercise, you can use the back of the sheet. If this is still not enough space, you can add one (or more) extra sheet(s) to the exam. In this case, indicate it clearly in the booklet, and write your name and student number on the top of that (these) sheet(s).
- Do not write on the table below.
- Sticking to the rules above is worth 10 points.
- It is absolutely NOT allowed to use calculator, phone, smartwatch, books, lecture notes or any other aids.
- Always give a short proof of your answer or a calculation to justify it, or clearly state the facts from the lecture notes you are using (unless it is stated explicitly in the question this is not needed.).

Exercise	Points	Score
1	12	
2	25	
3	17	
4	16	
5	20	
Follow the rules (name on each page...)	10	
Total	100	

Exercise 1 (12 pts)

Let X, Y, Z be independent and uniformly distributed over $(0,1)$. Compute $\mathbb{P}(X \geq YZ)$.

Solution: Since the random variables are independent,

$$f_{X,Y,Z}(x,y,z) = f_X(x)f_Y(y)f_Z(z) = 1, \qquad 0 \leq x \leq 1, \quad 0 \leq y \leq 1, \quad 0 \leq z \leq 1,$$

we have

$$\begin{aligned}
\mathbb{P}(X \geq YZ) &= \iiint_{x \geq yz} f_{X,Y,Z}(x,y,z)dx\,dy\,dz \\
&= \int_0^1 \int_0^1 \int_{yz}^1 dx\,dy\,dz \\
&= \int_0^1 \int_0^1 (1 - yz)dy\,dz \\
&= \int_0^1 (1 - \frac{z}{2})dy\,dz \\
&= \frac{3}{4}.
\end{aligned}$$

Exercise 2 (a:10, b:15 pts)

Let X and Y be independent Poisson random variables with respective means λ_1 and λ_2.

a) Prove that $X+Y$ is Poisson distributed with mean $\lambda_1+\lambda_2$.

b) Prove that the conditional distribution of X given that $X+Y=n$ is a binomial distribution and find the parameters of this binomial distribution.

Solution:

a) Since X and Y are independent, we have

$$M_{X+Y}(t) = M_X(t)M_Y(t).$$

And because they are Poisson distributed with means λ_1 and λ_2 we get

$$M_{X+Y}(t) = \exp(\lambda_1(e^t-1))\exp(\lambda_2(e^t-1)) = \exp((\lambda_1+\lambda_2)(e^t-1)).$$

Hence, $X+Y$ is Poisson distributed with mean $\lambda_1+\lambda_2$.

b)

$$\begin{aligned}
\mathbb{P}(X=k \mid X+Y=n) &= \frac{\mathbb{P}(X=k, X+Y=n)}{\mathbb{P}(X+Y=n)} \\
&= \frac{\mathbb{P}(X=k, Y=n-k)}{\mathbb{P}(X+Y=n)} \\
&= \frac{\mathbb{P}(X=k)\mathbb{P}(Y=n-k)}{\mathbb{P}(X+Y=n)} && \text{(by independence of } X \text{ and } Y\text{)} \\
&= \frac{e^{-\lambda_1}\lambda_1^k}{k!}\frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!}\left[\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}{n!}\right]^{-1} && \text{(by item a.)} \\
&= \frac{n!}{(n-k)!k!}\frac{\lambda_1^k\lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n} \\
&= \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k} \\
&= \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(1-\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{n-k}
\end{aligned}$$

In other words, the conditional distribution of X given that $X+Y=n$ is the binomial distribution with parameters n and $\lambda_1/(\lambda_1+\lambda_2)$.

Exercise 3 (17 pts) Along a road 1 mile long are 3 people "*distributed at random according to the continuous uniform distribution and independently of each other.*" Let $d \in (0, \frac{1}{2})$. Find the probability that no 2 people are less than a distance of d miles apart.

Solution: Denote by X_1, X_2 and X_3 the respective positions (in miles) of people 1, 2 and 3. For $\{i, j, k\} = \{1, 2, 3\}$, set

$$E_{ijk} = \{X_i + d < X_j \text{ and } X_j + d < X_k\}.$$

Note that there are $3! = 6$ ways to order the numbers $1, 2, 3$ so there are 6 such events.
Note that by symmetry these events all have the same probability to happen.
Note that these events are pairwise disjoint and that their union is exactly the event that no 2 people are less than a distance of d miles apart.
Hence we have

$$\begin{aligned}\mathbb{P}(\text{no 2 people are less than a distance of d miles apart}) &= \mathbb{P}(E_{123} + E_{132} + E_{213} + E_{231} + E_{312} + E_{321}) \\ &= \mathbb{P}(E_{123}) + \mathbb{P}(E_{132}) + \mathbb{P}(E_{213}) + \mathbb{P}(E_{231}) + \mathbb{P}(E_{312}) + \mathbb{P}(E_{321}) \\ &= 6 \cdot \mathbb{P}(E_{123}).\end{aligned}$$

Since the positions are independent and uniformly distributed we have that

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3) = f_{X_1}(x_1) f_{X_2}(x_2) f_{X_3}(x_3) = \mathbf{1}(x_1 \in [0,1])\mathbf{1}(x_2 \in [0,1])\mathbf{1}(x_3 \in [0,1]) = \mathbf{1}(x_1,x_2,x_3 \in [0,1]).$$

Therefore

$$\begin{aligned}&\mathbb{P}(\text{no 2 people are less than a distance of d miles apart}) \\ &= 6 \cdot \mathbb{P}(E_{123}) \\ &= 6 \int_0^1 \int_0^1 \int_0^1 \mathbf{1}(E_{123}) dx_3\, dx_2\, dx_1 \\ &= 6 \int_0^{1-2d} \int_{x_1+d}^{1-d} \int_{x_2+d}^{1} dx_3\, dx_2\, dx_1 \\ &= 6 \int_0^{1-2d} \int_{x_1+d}^{1-d} (1 - x_2 - d) dx_2\, dx_1 \\ &= 6 \int_0^{1-2d} \int_0^{1-2d-x_1} y_2 dy_2\, dx_1 && (\text{substitution } y_2 = 1 - d - x_2) \\ &= 3 \int_0^{1-2d} (1 - 2d - x_1)^2 dx_1 \\ &= 3 \int_0^{1-2d} y_1^2 dx_1 && (\text{substitution } y_1 = 1 - 2d - x_1) \\ &\boxed{= (1-2d)^3.}\end{aligned}$$

Exercise 4 (a:8, b:8 pts)

a. State Markov's inequality.

b. Let X be a random variable with expected value $\mu \in \mathbb{R}$ and standard deviation $\sigma \in (0, \infty)$. Use Markov's inequality to show that

$$\mathbb{P}(|X - \mu| \geq c) \leq \frac{\sigma^2}{c^2} \quad \text{for any } c > 0.$$

Remark: This inequality is called Chebyshev's inequality.

Solution:

a. Let $a > 0$ and Y be a non-negative random variable. Markov's inequality states that

$$\mathbb{P}(Y \geq a) \leq \frac{1}{a}\mathbb{E}[Y].$$

b. It follows from a by setting $Y = (X - \mu)^2$, $c = \sqrt{a}$. Indeed

$$\mathbb{P}(|X - \mu| \geq c) = \mathbb{P}((X - \mu)^2 \geq c^2) = \mathbb{P}(Y \geq a) \leq \frac{1}{a}\mathbb{E}[Y] = \frac{\sigma^2}{c^2}.$$

Exercise 5 (a:4, b:4, c:4, d:4, e:4 pts) A gang of 5 robbers stole 3 diamonds. For each of the following situations find how many ways there are to split the loot.
Remark: Give your answer (a number) directly in the box below. You are not required to provide any explanation in this exercise. For each question you get either zero or the full four points.

a. The diamonds are distinct from one another and each robber receives at most 1 diamond.

The number of possibilities to split the loot is:

b. The diamonds are distinct from one another. Any of the robbers might receive several diamonds.

The number of possibilities to split the loot is:

c. The diamonds are identical and they are given to distinct robbers.

The number of possibilities to split the loot is:

d. The diamonds are identical. One robber receives 2 diamonds and one other receive 1 diamond.

The number of possibilities to split the loot is:

e. The diamonds are identical. Any of the robbers might receive several diamonds.

The number of possibilities to split the loot is:

Solution:

a. $5 \times 4 \times 3 = 60$

b. $5^3 = 125$

c. $\binom{5}{3} = 10$

d. $5 \times 4 = 20$

e. $10 + 20 + 5 = 35.$